

Form Factors in $\mathcal{N} = 4$ Super Yang-Mills and Periodic Wilson Loops

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Abstract

We calculate form factors of half-BPS operators in $\mathcal{N} = 4$ super Yang-Mills theory at tree level and one loop using novel applications of recursion relations and unitarity. In particular, we determine the expression of the one-loop form factors with two scalars and an arbitrary number of positive-helicity gluons. These quantities resemble closely the MHV scattering amplitudes, including holomorphicity of the tree-level form factor, and the expansion in terms of two-mass easy box functions of the one-loop result. Next, we compare our result for these form factors to the calculation of a particular periodic Wilson loop at one loop, finding agreement. This suggests a novel duality relating form factors to periodic Wilson loops.

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1 Introduction

One of the important challenges ahead is that of relating two apparently disconnected realms, that of scattering amplitudes, and that of correlation functions. The first candidate theory to study is naturally maximally supersymmetric Yang-Mills theory (SYM). A step in this direction was taken in [1–3], where correlation functions of composite operators in $\mathcal{N} = 4$ SYM theory were considered in certain special lightlike limits where the distances between adjacent insertion points become null. Surprisingly, it was found that in the lightlike limit considered there is a simple relation between $(n + l)$ -point correlators with l insertions of the Lagrangian, and the integrand of the n -point MHV amplitude evaluated at l loops.

In this paper we will focus on an interesting class of physical observables which sit between the two worlds of amplitudes and correlation functions. These are the form factors, i.e. matrix elements of gauge-invariant, composite operators \mathcal{O} between the vacuum and some external scattering state,¹

$$F(1, \dots, n) = \langle 1, \dots, n | \mathcal{O}(0) | 0 \rangle . \quad (1.1)$$

The external state can in principle contain all different particles in the theory. At strong coupling, these objects have been considered recently in [4, 5], where it was found that their calculation is technically equivalent to that of a periodic Wilson loop whose contour is specified by the lightlike momenta of the scattered particles. In distinction to the amplitude calculation at strong coupling [6], the momenta do not sum to zero as there is an operator insertion carrying momentum $-q$; furthermore, the sum of the momenta, $\sum_{l=1}^n p_l = q$ is not null, $q^2 \neq 0$, and defines the period of the Wilson loop contour relevant at strong coupling. It would be interesting to see whether this Wilson loop/form factor connection can be extended to weak coupling as was the case for the Wilson loop/amplitude duality [6–8]. In this paper we will find some evidence for this at one loop.

At strong coupling, and leading order in $1/\sqrt{\lambda}$, the form factor calculation is insensitive to the polarisation of the external particles and, importantly, to the precise choice of the operator, as long as its anomalous dimension is small compared to $\sqrt{\lambda}$ [5]. This is similar to a feature of the amplitude calculation of [6], where the polarisations of the external states are only expected to play a role starting at one loop in the $1/\sqrt{\lambda}$ expansion. At weak coupling, we have no reason a priori to expect that the form factor will be independent of the choice of the operator. We will then focus on the simplest class of operators in $\mathcal{N} = 4$ SYM, that of half-BPS protected operators.

To be specific, we will consider the operator $\text{Tr}(\phi_{12}\phi_{12})$ inserted between a state

¹We note that the dimension of the form factor in (1.1) is equal to $d_{\mathcal{O}} - n$, where $d_{\mathcal{O}}$ is the physical dimension of \mathcal{O} .

containing two scalars and an arbitrary number of gluons with positive helicity. The simplest member of this class is constructed with an external state containing just the two scalars, and is the Sudakov form factor $F(q^2) := \langle \phi_{12}(p_1) \phi_{12}(p_2) | \text{Tr}(\phi_{12} \phi_{12})(0) | 0 \rangle$, where $q := p_1 + p_2$. This quantity was calculated at one and two loops in $\mathcal{N} = 4$ SYM in a pioneering paper [9] by van Neerven, using an approach based on unitarity applied to Feynman diagrams developed in [10]. Remarkably, the two-loop Sudakov form factor was found to satisfy an iterative relation very similar to that discovered in [11, 12] for the four-point MHV scattering amplitudes.²

We begin in Section 2 by rederiving the one-loop result for the Sudakov form factor with an elementary application of unitarity. In Section 3 we will move on to consider a more generic class of n -point form factors containing $n - 2$ positive-helicity gluons in addition to the two scalars. We will first determine their expression at tree level for arbitrary n using recursion relations. Our result, Equation (3.3), is a holomorphic function of the spinor variables describing the particle momenta, and is a close relative of the Parke-Taylor MHV scattering amplitude. For example, it localises on a complex line in twistor space, as was found in [13] for the MHV amplitude.

We will then address the one-loop calculation of this class of form factors constructed from a BPS operator and a state containing two scalars and an arbitrary number of gluons. Our strategy will consist in applying unitarity directly at the level of the form factor, thus bypassing Feynman diagrams, in the same spirit as the unitarity-based approach of [14, 15] for scattering amplitudes. The quantities entering the form factor cuts are tree-level form factors and amplitudes. Both are gauge invariant objects, whose compact expressions we will recycle inside the loops, thereby obtaining simple one-loop integrands. The final result for this class of form factors at one loop is remarkably simple – see Equation (3.17). It is also very reminiscent of the expression for an n -point MHV amplitude at one loop, as we will describe.

This close similarity will serve as an inspiration for Section 4, where we compare our weak-coupling result (3.17) for the form factors to a one-loop calculation of the periodic Wilson loop which, at strong coupling, computes the same quantity. Quite surprisingly, we will find that this periodic Wilson loop calculates the form factor also at weak coupling, or more precisely, the ratio of the one-loop form factor to its tree-level expression. Some issues related to gauge invariance of the prescription will also be discussed.

Finally, we present our conclusions and discuss directions for further research in Section 5. A few appendices complete the paper. In particular, in Appendix A we write down a BCFW recursion relation [16, 17] for the n -point tree-level form factor discussed earlier. In Appendix B, we review for the reader's convenience the iterative

²We review this iterative structure of the form factor in Appendix B.

structure for the Sudakov form factor in $\mathcal{N} = 4$ SYM found in [9]. Appendix C contains the expression of the finite two-mass easy box functions, which appear in the form factor and Wilson loop calculations.

2 The Sudakov form factor

We begin by introducing the operator

$$\mathcal{O}_{ABCD} := \text{Tr}(\phi_{AB}\phi_{CD}) - \frac{1}{12}\epsilon_{ABCD}\text{Tr}(\bar{\phi}^{LM}\phi_{LM}) , \quad (2.1)$$

where $\bar{\phi}^{AB} := (1/2)\epsilon^{ABCD}\phi_{CD}$. The operator in (2.1) belongs to the $\mathbf{20}'$ representation of the $SU(4)$ R -symmetry group and is half BPS, i.e. it has vanishing anomalous dimension. Without loss of generality we will focus in the rest of this paper on its particular component

$$\mathcal{O} := \text{Tr}(\phi_{12}\phi_{12}) . \quad (2.2)$$

In this section we concentrate on the simplest form factor one can construct using this operator, namely the two-point or Sudakov form factor,

$$F(q^2) := \langle \phi_{12}(p_1)\phi_{12}(p_2) | \mathcal{O}(0) | 0 \rangle , \quad (2.3)$$

where

$$q := p_1 + p_2 . \quad (2.4)$$

As mentioned in the Introduction, this form factor in $\mathcal{N} = 4$ SYM was studied at one and two loops in [9], and calculated in [12] up to three loops. Here we will reproduce the one-loop result of [9] as a simple application of unitarity. We have also performed a similar calculation at two loops and found agreement with the result of [9], but we will not discuss it here.

2.1 The one-loop Sudakov factor from unitarity cuts

The Sudakov form factor is very simple to compute since, by Lorentz invariance, (2.3) depends only on q^2 ; it can then be determined entirely from its unitarity cut in the q^2 channel. In [9, 10] this procedure was carried out at one and two loops; here we depart from the approach of these two references in that we apply unitarity directly at the level of the form factor, bypassing Feynman diagrams.

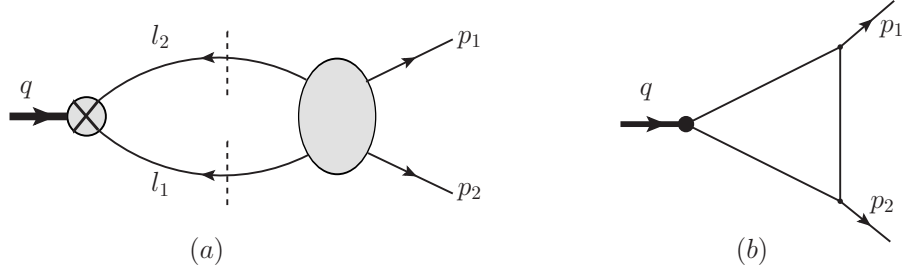


Figure 1: In Figure (a) we show the diagram calculating the cut in the q^2 -channel of the Sudakov form factor (2.3). The cross denotes a form factor insertion. A second diagram with legs 1 and 2 swapped has to be added and doubles up the result of the first diagram. The result of this cut is given by (twice) a cut one-mass triangle function, depicted in Figure (b).

The q^2 -cut of the form factor (i.e. its discontinuity in the q^2 -channel) is obtained from the diagram on the left-hand side of Figure 1, whose expression is³

$$F^{(1)}(q^2)|_{q^2\text{-cut}} = 2 \int d\text{LIPS}(l_1, l_2; q) F^{(0)}(l_1, l_2; q) A^{(0)}(\phi_{12}(p_1), \phi_{12}(p_2), \phi_{34}(l_1), \phi_{34}(l_2)) , \quad (2.5)$$

where the Lorentz invariant phase space measure is

$$d\text{LIPS}(l_1, l_2; q) := d^D l_1 d^D l_2 \delta^+(l_1^2) \delta^+(l_2^2) \delta^D(l_1 + l_2 + q) , \quad (2.6)$$

and q is given in (2.4). The tree-level component amplitude appearing in (2.5), $A^{(0)}(\phi_{12}(p_1), \phi_{12}(p_2), \phi_{34}(l_1), \phi_{34}(l_2))$, can be extracted from Nair's superamplitude [18]

$$\mathcal{A}_{\text{MHV}} := g^{n-2} (2\pi)^4 \delta^{(4)} \left(\sum_{i=1}^n \lambda_i \tilde{\lambda}_i \right) \delta^{(8)} \left(\sum_{i=1}^n \lambda_i \eta_i \right) \prod_{i=1}^n \frac{1}{\langle ii+1 \rangle} , \quad (2.7)$$

where $\lambda_{n+1} \equiv \lambda_1$. The result is

$$A^{(0)}(\phi_{12}(p_1), \phi_{12}(p_2), \phi_{34}(l_1), \phi_{34}(l_2)) = \frac{\langle l_1 l_2 \rangle \langle 12 \rangle}{\langle l_2 1 \rangle \langle 2 l_1 \rangle} . \quad (2.8)$$

The other quantity appearing in (2.5), $F^{(0)}$ is the tree-level expression for the form factor (2.3), which is trivially equal to 1. Thus, we get

$$F^{(1)}(q^2)|_{q^2\text{-cut}} = 2 \int d\text{LIPS}(l_1, l_2; q) \frac{\langle 12 \rangle \langle l_1 l_2 \rangle}{\langle 2 l_1 \rangle \langle l_2 1 \rangle} = -2 q^2 \int d\text{LIPS}(l_1, l_2; q) \frac{1}{(l_2 + p_1)^2} . \quad (2.9)$$

³In this and the following formulae we omit a power of the 't Hooft coupling, defined as $a := (g^2 N)/(16\pi^2)(4\pi e^{-\gamma})^\epsilon$. Note that this is 1/2 the 't Hooft coupling defined in [12].

The expression in (2.9) represents the cut of a one-mass triangle $T^{1m}(q^2, \epsilon)$, depicted in Figure 1(b). It can instantly be lifted to a full loop integral since it depends only on a single kinematic invariant, much in the same spirit as [19]. Doing so we get

$$F^{(1)}(q^2) = 2 \text{Tri}(q^2, \epsilon) , \quad (2.10)$$

where the function $\text{Tri}(q^2, \epsilon) := q^2 T^{1m}(q^2, \epsilon)$ is explicitly evaluated in (B.3). This result agrees with that of [9].

3 Multi-point form factors

With the operator \mathcal{O} introduced in (2.2), we can now construct an infinite sequence of n -point form factors,

$$F(1, \dots, i_{\phi_{12}}, \dots, j_{\phi_{12}}, \dots, n; q) := \langle g^+(p_1) \cdots \phi_{12}(p_i) \cdots \phi_{12}(p_j) \cdots g^+(p_n) | \mathcal{O}(0) | 0 \rangle , \quad (3.1)$$

i.e. we take matrix elements of \mathcal{O} between the vacuum and a state containing the same two scalars already appearing in (2.3), along with $n - 2$ positive-helicity gluons. The particular form factor we consider is colour-ordered with respect to the positions of the external particles. However notice that the operator \mathcal{O} is a colour singlet, hence the momentum q it carries can be inserted at any position in colour ordering. We now present the calculation of (3.1), first at tree level and then at one loop.

3.1 Tree level

At tree level, it is easy to calculate the form factor (3.1). Indeed, we observe that factorisation theorems are valid also for form factors – in fact they apply to Green’s function in general, see for example [20] for a discussion. We can then use tree-level factorisation in order to write down a BCFW recursion relation [16, 17] for form factors. Our result for this quantity is very simple,⁴

$$F^{(0)}(1, \dots, i_{\phi_{12}}, \dots, j_{\phi_{12}}, \dots, n; q) = g^{n-2} (2\pi)^4 \delta^{(4)} \left(\sum_{k=1}^n \lambda_k \tilde{\lambda}_k - q \right) F_{\text{MHV}} , \quad (3.2)$$

where

$$F_{\text{MHV}} = \frac{\langle ij \rangle^2}{\langle 12 \rangle \cdots \langle n1 \rangle} . \quad (3.3)$$

⁴The calculation of (3.2) is presented in Appendix A. We have also checked our result against Feynman diagrams in a few cases.

Here $p_m := \lambda_m \tilde{\lambda}_m$, and $\sum_{m=1}^n p_m := q$ is the momentum carried by the operator insertion. A number of remarks are in order.

1. The expression in (3.3) is purely holomorphic in the spinor variables of the external particles. In this sense this form factor is the closest off-shell relative of the Parke-Taylor MHV amplitude. We will refer to this form factor as to the “MHV form factor”. Note however that the momenta do not sum to zero. We also note that this expression is very reminiscent of the formula for the infinite sequence of Higgs + gluons amplitudes considered in [21].

2. We can easily transform the tree-level form factor (3.2) to Penrose’s twistor space, with the result

$$F_{\text{MHV}} \int d^4x \, e^{iqx} \prod_{m=1}^n \delta^{(2)}(\mu_m + x\lambda_m) , \quad (3.4)$$

where (λ_m, μ_m) are the twistor space coordinates of the m^{th} particle. As for the case of the MHV amplitude, holomorphicity of the tree-level form factor (3.3) ensures that F_{MHV} can be pulled out of the half-Fourier transform. Equation (3.4) shows that our MHV form factor is localised on a line in twistor space, similarly to the MHV amplitude [13].

3. We notice that (3.3) satisfies an auxiliary (helicity) condition

$$\frac{1}{2} \left(-\lambda_m \frac{\partial}{\partial \lambda_m} + \tilde{\lambda}_m \frac{\partial}{\partial \tilde{\lambda}_m} \right) F(1, \dots, n) = h_m F(1, \dots, n) , \quad (3.5)$$

where h_m is the helicity of the m^{th} particle. This relation is just the statement that each external state must transform appropriately under the little group of a massless vector [13].

3.2 The one-loop MHV form factor from unitarity

We can now insert the compact expressions for the tree-level form factor (3.3) found earlier into unitarity cuts, and glue it with tree-level amplitudes in order to build cuts of loop form factors. Our strategy will consist in calculating the cuts of form factors in all kinematic channels, and then reconstructing the function which has all the correct cuts, in complete analogy with the approach of [14, 15] for amplitudes. The final result will be expressed in terms of the elements of the basis of one-loop scalar integral functions, times rational coefficients.⁵

⁵This approach might of course miss purely rational terms, not linked to discontinuities of integral functions. Since we work in $\mathcal{N} = 4$ super Yang-Mills such terms are absent.

The operator we have chosen in (2.1) is unrenormalised; as a consequence, the form factor will contain no ultraviolet-divergent functions. At one loop this implies the absence of bubbles. Unlike amplitudes, we will encounter one-mass triangles; however, as we shall see momentarily, their purpose will be that of canceling certain otherwise unwanted infrared divergences in multi-particle channels. Indeed, anticipating our story a little, the result of our calculation will consist of a sum of infrared-divergent terms containing only two-particle kinematic invariants, along with a sum over all possible finite parts of two-mass easy box functions.

We now discuss all possible cuts of the form factor. Here is an outline of the main features of our calculation:

1. We will perform cuts in three distinct channels. The first one is the q^2 -channel, where the tree-level form factor entering the cut is a Sudakov form factor. In the second case, which needs to be treated separately, a tree form factor with one additional external particle enters the cut. These two cases give rise to two-mass easy box functions, as well as to triangles. Finally, in the most generic channel, the form factor entering the cut expression has an arbitrary number of legs. Importantly, in this case we will discover that, after factoring out the tree-level expression of the form factor, the integrand will be the same as that of the one-loop MHV amplitude in $\mathcal{N} = 4$ super Yang-Mills, calculated in [14].

2. In principle we will have to perform sums over all possible internal states which can run in the loop. We have found that, after performing the sums over different states when necessary, the cut-integrand becomes the same in all channels. This is in complete analogy with the calculation of [14] of the one-loop MHV scattering amplitude in $\mathcal{N} = 4$ SYM.

3. The form factor is ordered with respect to the position of the external particles in colour space. One may be tempted to think of the form factor as an amplitude with one (or more, if there are more operator insertions) leg going off shell, but this picture would not be true as far as the colour ordering is concerned since the operator insertion is a colour singlet. Hence, its position – the position of the “off-shell line” – does not affect the colour ordering of the external states. One must therefore insert the operator in all possible ways and sum over the corresponding contributions. In practice, this possibility will arise only in the q^2 -cut diagrams.

3.2.1 The q^2 channel

We begin by computing the cut in the q^2 channel, where

$$q = \sum_{k=1}^n p_k , \quad (3.6)$$

is the sum of the momenta of the external particles. This cut is represented in Figure 2, and is given explicitly by

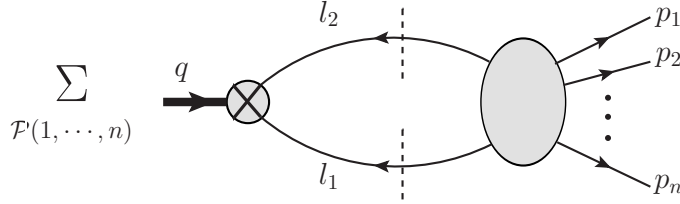


Figure 2: *The q^2 -cut of the one-loop form factor. Note that the complete cut is obtained by summing over cyclic permutations of $(1, \dots, n)$.*

$$F^{(1)}(1, \dots, n; q) \big|_{q^2\text{-cut}} = \int d\text{LIPS}(l_1, l_2; q) \sum_{\mathcal{P}(1, \dots, n)} \left\{ F^{(0)}(l_1, l_2; q) A^{(0)}(\phi_{34}(l_1), \phi_{34}(l_2), g^+(p_1), \dots, \phi_{12}(p_i), \dots, \phi_{12}(p_j), \dots, g^+(p_n)) \right\} . \quad (3.7)$$

Note that we have included a sum over cyclic permutations of the particles $1, 2, \dots, n$. This is because the insertion of the operator does not affect the colour ordering, therefore all terms in the sum have the same ordering in colour space.

We now move on to describe the ingredients appearing in (3.7). The tree-level form factor $F^{(0)}$ is equal to 1. The tree amplitude $A^{(0)}$, obtained from (2.7), is

$$A^{(0)} = \frac{\langle ij \rangle^2 \langle l_1 l_2 \rangle^2}{\langle l_2 1 \rangle \langle 12 \rangle \cdots \langle n l_1 \rangle \langle l_1 l_2 \rangle} . \quad (3.8)$$

Let us focus on one term in the sum, specifically that with the ordering $l_1, l_2, 1, \dots, n$ of the legs. The others will be obtained by cyclically permuting $1, \dots, n$. This term can be written as

$$F^{(0)}(1, \dots, n; q) \int d\text{LIPS}(l_1, l_2; q) \frac{\langle l_1 l_2 \rangle \langle n 1 \rangle}{\langle l_2 1 \rangle \langle n l_1 \rangle} ,$$

where $F^{(0)}(1, \dots, n; q)$ is the tree-level MHV form factor (3.3). It is easy to show that

$$\begin{aligned} \frac{\langle l_1 l_2 \rangle \langle n 1 \rangle}{\langle l_2 1 \rangle \langle n l_1 \rangle} &= \frac{2 \left[(p_1 q)(l_1 p_n) - (p_1 l_1)(q p_n) + (p_1 p_n)(q l_1) \right]}{(l_2 + p_1)^2 (l_1 + p_n)^2} \\ &= \frac{2(q p_n)(q p_1) - q^2(p_1 p_n)}{(l_2 + p_1)^2 (l_1 + p_n)^2} + \frac{(p_1 q)}{(l_2 + p_1)^2} + \frac{(q p_n)}{(l_1 + p_n)^2} , \end{aligned} \quad (3.9)$$

where $(p_a p_b) := p_a \cdot p_b$. This reduction thus leads to the q^2 -cut integral of a two-mass easy box and two scalar triangles, all of which can be calculated using standard formulae, see [9] as well as [22–26] for more recent applications.⁶ The two-mass easy box obtained here has massless legs p_1 and p_n , whereas the momenta of the massive corners are $-q$ and $P_{2n-1} = q - p_n - p_1$, where

$$P_{ab} := p_a + \dots + p_b . \quad (3.10)$$

Furthermore, we have two two-mass triangles. The first one has massless leg p_1 and massive legs $-q$ and $q - p_1$; for the second we just replace p_1 by p_n . Notice that by summing over all permutations we will generate twice every possible two-mass triangle, where the massless leg is in turn any one of the momenta of the particles, and one of the massive corners has momentum $-q$. Furthermore, we will produce all possible two-mass easy boxes with massless legs $a - 1$ and a , and massive legs $-q$ and $P_{a+1 a-1}$ for any $a \in (1, \dots, n)$.

Rather than calculating these dispersion integrals, we will now proceed to inspect other kinematic channels. This will enable us to reconstruct the one-loop form factor from its cuts.

3.2.2 The cut in a generic kinematic channel

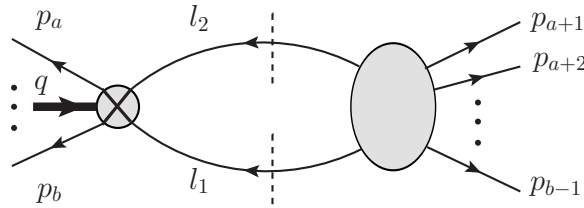


Figure 3: *The cut of the one-loop form factor in the $s_{a+1, b-1}$ -channel.*

Here we discuss the most generic cut, depicted in Figure 3. There are three possibilities to consider. The two scalars can either be emitted both from the form factor; or

⁶The important observation that dimensionally regularised dispersion integrals are well-defined, and rather simple objects to compute, was made in [10].

both from the MHV amplitude; finally, one can be emitted from the form factor, and one from the amplitude. In the latter case, it is necessary to sum over gluons and fermions, which can propagate in the loop. Remarkably, the three cases give rise to the same integrand. For this reason, in Figure 3 we need not distinguish the different types of particles.

For the sake of definiteness, we focus on the case where two scalars are emitted from the amplitude. The corresponding expression is quickly seen to be

$$F^{(1)}|_{s_{a+1,b-2}\text{-cut}} = \int d\text{LIPS}(l_1, l_2; P_{a+1,b-1}) \quad (3.11)$$

$$F^{(0)}(-l_2^\phi, -l_1^\phi, b^{g^+}, \dots, a^{g^+}; q)$$

$$A^{(0)}(l_1^\phi, l_2^\phi, (a+1)^{g^+}, \dots, i^\phi, \dots, j^\phi, \dots, (b-1)^{g^+}),$$

where P_{ab} is defined in (3.10), and the superscript indicates the particle's species.

Notice that the particles emitted from the tree-level form factor fix the colour ordering, hence there is no sum over permutations to be performed in this case. Using the explicit expressions of the tree-level form factor and the amplitude, (3.11) becomes

$$F^{(1)}|_{s_{a+1,b-2}\text{-cut}} = F^{(0)} \int d\text{LIPS}(l_1, l_2; s_{a+1,b-1}) \frac{\langle a a+1 \rangle \langle l_2 l_1 \rangle}{\langle a l_2 \rangle \langle l_2 a+1 \rangle} \frac{\langle b-1 b \rangle \langle l_1 l_2 \rangle}{\langle b-1 l_1 \rangle \langle l_1 b \rangle}, \quad (3.12)$$

where $F^{(0)}$ is now the tree-level form factor (3.3). The reader may have recognised that the integrand of (3.12) is the same as that appearing in the two-particle cuts of the one-loop MHV amplitude considered in [14]. Performing the reduction as in that paper, one finds that the integrand of (3.12) can be recast as the sum of four terms,

$$R(b, a+1) + R(b-1, a) - R(b, a) - R(b-1, a+1), \quad (3.13)$$

where

$$R(b, a) := \frac{\langle b l_2 \rangle \langle a l_1 \rangle}{\langle b l_1 \rangle \langle a l_2 \rangle} = \frac{2[(l_1 p_b)(l_2 p_a) + (l_1 p_a)(l_2 p_b) - (l_1 l_2)(p_a p_b)]}{2(l_1 p_b) 2(l_2 p_a)}$$

$$= 1 + \frac{(p_b P)}{2(p_b l_1)} + \frac{(p_a P)}{2(p_a l_2)} + \frac{2(p_a P)(p_b P) - P^2(p_a p_b)}{2(p_b l_1)2(p_a l_2)}, \quad (3.14)$$

and we used momentum conservation $l_1 + l_2 + P = 0$, where P is the cut momentum ($P^2 = s_{a+1,b-1}$ in this cut). We also used the cut conditions $l_1^2 = l_2^2 = 0$.

Equation (3.14) shows that each R -function gives rise to a bubble, two triangles and a two-mass easy box function. When all four of the terms in (3.13) are present, bubbles and triangles cancel. We will see in the next section that when $b = a$ there are

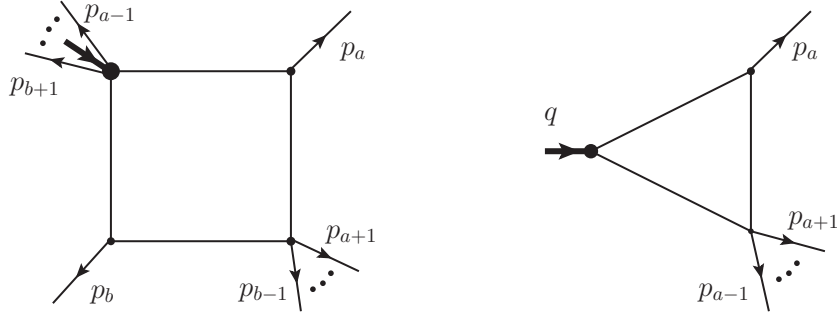


Figure 4: *On the left, we represent a two-mass easy box function. The momenta p_a and p_b are null, whereas, in general the remaining momenta $P := P_{a+1b-1}$ and $Q := -q + P_{b+1a-1}$ are not null. The cases when either P^2 or Q^2 , or both, are also null, correspond to the one-mass and zero-mass boxes, obtained as smooth limits from the expression (C.1) of the two-mass box function. On the right, we represent a two-mass triangle arising from the cuts considered in Section 3.2.3. The thick line represents the momentum carried by the operator.*

some surviving triangle functions (whereas bubble always cancel among themselves, as anticipated).

Specifically, the box function arising from $R(b, a)$ has massless legs a and b , and massive legs P_{a+1b-1} , and $-q + P_{b+1a-1}$. Furthermore, it appears here in the corner cut $s_{a+1,b-1}$, i.e. the cut momentum legs are those adjacent to the corner momentum P_{a+1b-1} . The other R -terms in (3.13) will give the same $s_{a+1,b-1}$ -cut of different two-mass easy box functions, where the entries l and m of $R(l, m)$ denote the massless legs of the box. Note that the same box in (3.14) will appear in all of its other cuts, with precisely the same coefficient. To see this, notice that the quantity $2(p_a P)(p_b P) - P^2(p_a p_b)$ is invariant under $P \rightarrow P + \alpha p_a + \beta p_b$ for any α and β . We remind the reader of the pictorial representations of the box and triangle functions in Figure 4.

3.2.3 The cut in a $(q - p_a)^2$ channel

Next we consider a channel where a single particle, of momentum p_a , is emitted from the form factor – see Figure 5. This $(q - p_a)^2$ -channel cut can be regarded as the special $a = b$ case of the generic $s_{a+1,b-1}$ -channel cut described in the previous section. Note that in the corresponding calculation of the one-loop MHV amplitude, with a single particle of momentum p_a emitted from a vertex, this cut would be absent since $p_a^2 = 0$. However, the momentum flowing from the form factor is here $-q + p_a$, which is not null, and there is a new nontrivial cut to compute.

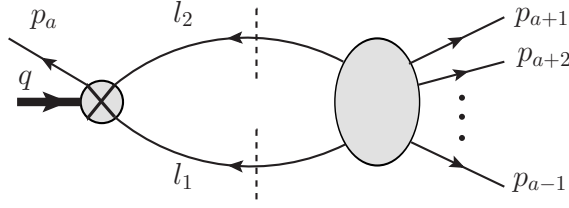


Figure 5: *The $(q - p_a)^2$ -cut of the one-loop form factor.*

The cut integral is given by a formula identical to (3.12) except that now $a = b$. The result for the cut is therefore

$$R(a, a + 1) + R(a - 1, a) - R(a, a) - R(a - 1, a + 1) , \quad (3.15)$$

where, using (3.14), we see that $R(a, a) = 1$. As a consequence, bubbles still cancel among the four terms, but there are uncanceled triangles (as well as boxes). Specifically, using the second line of (3.14), it is immediate to see that there is a single surviving triangle, which gives a contribution

$$2 \frac{(p_a P)}{2(p_a l_1)} , \quad (3.16)$$

where in this particular cut, $P = -q + p_a = -P_{a+1 a-1}$. This is therefore a two-mass triangle, where the massless leg has momentum p_a , and the two massive legs have momentum $P_{a+1 a-1}$ and $-q$, respectively. We have thus found the same set of two-mass triangles as we found from the q^2 -cuts discussed earlier.

3.3 The complete result

The next step consists in inspecting the integral functions appearing in all cuts; one then reconstructs the function which has the correct cuts in all channels. An important observation is that the triangle functions (with the precise coefficient they appear with) precisely cancel all infrared-divergent terms in the q^2 and $(q - p_a)^2$ multi-particle channels. The remaining multi-particle infrared-divergent terms cancel among all the boxes in the same way as in the calculation of the one-loop MHV amplitude.

After these cancellations are performed, the final result (divided by the tree-level form factor) has a remarkably simple form: it consists of the sum of infrared-divergent terms containing only two-particle invariants made of momenta which are adjacent in colour space, plus a sum of finite parts of two-mass easy box functions, where the

massless legs are any two of the particle's momenta, all with coefficients equal to one.⁷

$$F^{(1)}(1, \dots, n; q) = F^{(0)}(1, \dots, n; q) \left[- \sum_{l=1}^n \frac{(-s_{ll+1})^{-\epsilon}}{\epsilon^2} + \sum_{a,b} \text{Fin}^{2\text{me}}(p_a, p_b, P, Q) \right]. \quad (3.17)$$

In the sum on the right-hand side of (3.17) the external particles are distributed following cyclic ordering. Note however that the operator insertion, carrying momentum $-q$, does not affect this ordering.

As an example, we list below the finite box functions appearing in the four-point result: $(1, \{-q\}, 2, \{3, 4\})$ (massless legs 1 and 2); $(1, 2, 3, \{4, -q\})$, $(1, \{2, -q\}, 3, 4)$ (massless legs 1 and 3); $(1, \{2, 3\}, 4, \{-q\})$ (massless legs 1 and 4); $(2, \{-q\}, 3, \{4, 1\})$ (massless legs 2 and 3); $(2, 3, 4, \{1, -q\})$, $(2, \{3, -q\}, 4, 1)$ (massless legs 2 and 4); $(3, \{-q\}, 4, \{1, 2\})$ (massless legs 3 and 4). Here we denote by $(a, \{P\}, b, \{Q\})$ a two-mass easy box with massless momenta p_a and p_b , and corner momenta P and Q .

4 Comparing form factors to periodic Wilson loops

In [4,5], a prescription to calculate form factors at strong coupling using the AdS/CFT correspondence was proposed. We recall that the form factor calculation, as well as the scattering amplitude calculation at strong coupling [4,6,27–30], are both equivalent to the problem of computing minimal surfaces in AdS space, and hence Wilson loops, but the boundary conditions in the two cases are quite different. For n -point amplitudes, the boundary (the contour of the Wilson loop) is the n -edged closed polygon obtained by joining the lightlike momenta of the particles following the order induced by the colour structure of the planar amplitude. For n -point form factors, the boundary is an infinite periodic sequence of n lightlike segments [4,5], see Figure 6 for an example of a form factor with three particles, of momenta p_1 , p_2 and p_3 . The period q is the momentum of the inserted operator, which at strong coupling corresponds to having an additional closed string state inserted on the worldsheet. Therefore, in the form factor case the worldsheet stretches all the way from the boundary at $r = 0$ to $r \rightarrow \infty$, where r is the radial coordinate of the T-dual AdS space [4,5].

It was observed in [7,8] that the one-loop calculation of a Wilson loop with the same polygonal contour as in the strong-coupling calculation reproduces the one-loop MHV amplitude, divided by the tree amplitude. This was the first manifestation of a

⁷A more explicit way to write the sum of finite two-mass easy box functions in (3.17) is $\sum_{i=1}^n \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor - 1} \left(1 - (1/2)\delta_{n/2-1,r} \right) \text{Fin}_{n;r;i}^{2\text{me}}$, where the relation to the functions $\text{Fin}^{2\text{me}}(p_a, p_b, P, Q)$ depicted in Figure 4 is obtained by setting $p_a = p_{i-1}$, $p_b = p_{i+r}$, and $P = p_i + \dots + p_{i+r-1}$. We give the explicit expression of these finite box functions in Appendix C.

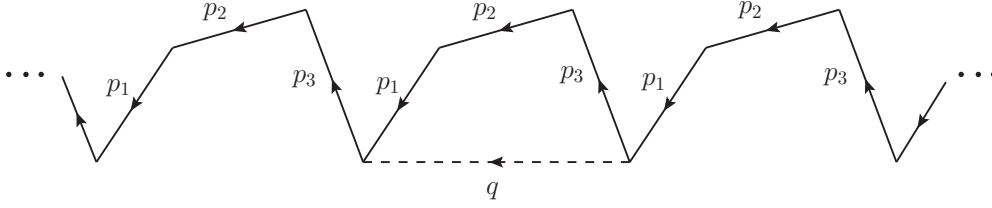


Figure 6: *Periodic Wilson loop featuring in the duality with form factors. In this case the period is $q = p_1 + p_2 + p_3$.*

new amplitude/Wilson loop duality at weak coupling, which was further checked and studied in [31–42]. An immediate question is therefore whether there is also a form factor/periodic Wilson loop duality at weak coupling, i.e. whether the form factor can be equivalently evaluated at weak coupling from a Wilson loop with the same periodic contour found at strong coupling. In this section, we calculate the periodic Wilson loop at one loop, and compare it to the result (3.17) we obtained earlier for the n -point form factor using unitarity. We will find that the two are in agreement.

The Wilson loop we consider is

$$W[\mathcal{C}_n] := \text{Tr} \mathcal{P} \exp \left[ig \int_{\mathcal{C}_n} d\tau \dot{x}^\mu(\tau) A_\mu(x(\tau)) \right], \quad (4.1)$$

where an example of a periodic contour \mathcal{C}_n with $n = 3$ is depicted in Figure 6. For the calculation it is convenient to choose the Feynman gauge and, as discussed in [7, 8], this leads at one loop to two classes of diagrams, namely the infrared-divergent cusp diagrams, where a propagator connects two adjacent edges, and those diagrams where the two edges connected by a propagator are not adjacent. Diagrams in the latter class are finite.

The first evidence of the duality is straightforward to detect, and comes from the infrared-divergent part. The form factor result (3.17) shows that the only divergent terms contain two-particle kinematic invariants made of momenta that are adjacent in colour ordering,

$$F^{(1)}(1, \dots, n; q) |_{\text{IR}} = F^{(0)}(1, \dots, n; q) \left[-\frac{1}{\epsilon^2} \sum_{i=1}^n (-s_{ii+1})^{-\epsilon} \right], \quad (4.2)$$

with $s_{ii+1} := (p_i + p_{i+1})^2$. This contribution exactly matches the sum of the one-loop cusp diagrams for the Wilson loop, in an identical way as for the matching of the infrared-divergent parts of one-loop MHV amplitudes and Wilson loop [7, 8].⁸ As in that case, the dual Wilson loop calculation does not reproduce the tree-level

⁸We recall that a cusp contributes a term $-(-s_{ii+1})^{-\epsilon}/\epsilon^2$ [7, 8, 43].

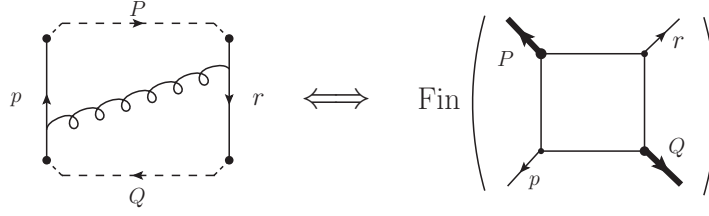


Figure 7: We illustrate in this figure the correspondence between Wilson loop diagrams connecting non-adjacent edges and finite parts of two-mass easy box functions. Notice that this correspondence holds in the Feynman gauge.

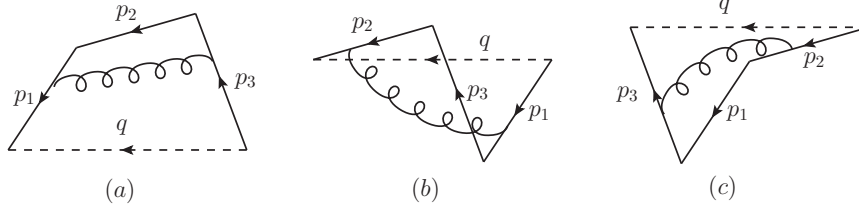


Figure 8: One-loop finite Wilson loop diagrams in the three-point case. Diagrams (a), (b), (c) are equal to the finite parts of the two-mass easy boxes $(1, 2, 3, \{-q\})$, $(2, 3, 1, \{-q\})$, and $(2, \{-q\}, 3, 1)$, respectively.

prefactor. Note that each period contains n cusps, and we sum once over these n cusps. In particular, for the two-point case, the period contains two cusps which give an identical contribution. This is related to the factor of two in the Sudakov form factor [44–51] in (2.10).

Next, we move to the finite parts. We found in (3.17) that the one-loop form factor contains a sum of finite parts of two-mass easy box functions (and one-mass boxes, as special cases of the former). It was shown in [8] that a Wilson loop diagram where a propagator connects two non-adjacent edges, called p and r on the left-hand side of Figure 7, is identical to the finite part of a two-mass easy box function with massless legs p and r , right-hand side of the same figure. The massive corners P and Q of the two-mass easy box are mapped to sums of adjacent momenta on the Wilson loop side, as shown in Figure 7. This crucial observation puts in one-to-one correspondence the finite boxes appearing in (3.17) with all the one-loop Wilson loop diagrams connecting non-adjacent edges in one period. For example, the three-point form factor contains the finite parts of three two-mass-easy boxes $(1, 2, 3, \{-q\})$, $(2, 3, 1, \{-q\})$, and $(2, \{-q\}, 3, 1)$. These functions are exactly given by the three Wilson loop diagrams depicted in Figure 8. It is not difficult to see that this mapping is true for general n -point form factors.

A few remarks are in order here.

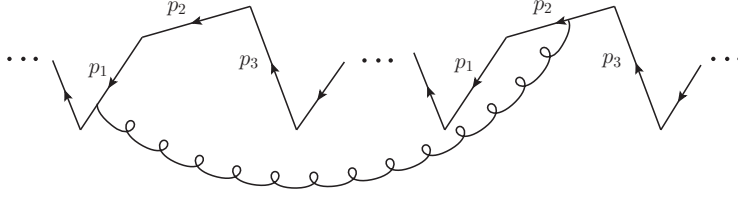


Figure 9: *In this figure we represent a one-loop Wilson loop diagram that should not be included.*

1. Notice that q is not part of the contour, and is therefore not connected to any other edge by propagators.

2. We draw all finite diagrams where we connect all pairs of non-adjacent legs within a period. We can in all cases map these configurations to finite two-mass easy box functions using the correspondence of Figure 7.

3. In principle one may think that the full calculation of a periodic Wilson loop would require a summation over all translationally inequivalent diagrams. In particular one should also include diagrams where a propagator stretches over more than one period, as for example that in Figure 9. As we have seen, in order to reproduce the form factor result these diagrams should not be included. This means that we need to make a truncation of the periodic Wilson loop and only consider the diagrams within one period. Notice that at strong coupling the form factor is calculated by the area of one period [4, 5].

4. By restricting to one period, we have succeeded in mapping the truncated periodic Wilson loop calculation, which is not obviously gauge invariant, to the form factor, which is gauge invariant by definition. It would be important to explore this issue further, in particular through higher-loop calculations of the periodic Wilson loop. We leave this for future work.

5. An important difference between the amplitude and the form factor calculations at strong coupling is the presence in the latter case of additional boundary conditions at $r \rightarrow \infty$ due to the operator insertion on the worldsheet. It would be important to understand what modifications of the weak-coupling Wilson loop calculation this entails. This might require the insertion of additional operators compensating the gauge non-invariance of the calculation.⁹

⁹We thank Gregory Korchemsky for discussions on the issue of gauge invariance.

5 Conclusions

To summarise, we have calculated form factors of half-BPS operators in $\mathcal{N} = 4$ super Yang-Mills theory at tree level and one loop. This took advantage of the powerful methods of recursion relations [16,17] and unitarity [14,15], which may also be applied to form factors. The expressions we found for the one-loop form factors with two scalars and an arbitrary number of positive-helicity gluons in particular share salient features with MHV scattering amplitudes. We then found that these form factors could be straightforwardly derived from periodic Wilson loops, suggesting that there is a new duality between form factors and periodic Wilson loops.

One can also easily generalise the form factor defined in (3.1) to one containing external states with different particles, in particular negative-helicity gluons. These should correspond to non-MHV extensions of the form factor we considered. It would be useful to explore a supersymmetric formulation of such quantities. Most importantly, it would be very interesting to consider form factors with insertions of different operators, including non-BPS ones such as the Konishi operator.

We also found that the tree-level form factors studied here localise in twistor space, and a similar question arises as to whether more complicated form factors (e.g. with insertions of the Konishi operator) enjoy similar localisation properties in twistor space. We anticipate that form factors similar to (3.1), but where the external states also contain negative-helicity gluons, will be localised on unions of lines as found in [13] for the tree-level non-MHV scattering amplitudes. One would also like to investigate the application of the whole MHV diagram approach to form factors, much in the same spirit as was done in [21] when considering interactions with massive bosons.

Finally, an obvious important question is to investigate this conjectured form factor/Wilson loop duality at higher loops. It seems likely that there will be many further interesting and productive directions to explore as part of applying the techniques and insights obtained from the study of scattering amplitudes to correlation functions more generally.

Acknowledgements

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Sokatchev for bringing the paper [9] to our attention. WJS was supported by a Leverhulme Research Fellowship. This work was supported by the STFC under a Rolling Grant ST/G000565/1.

A Recursion relations for tree-level form factors

In this appendix, we apply the BCFW recursion relation to calculate the tree-level form factor defined in (2.3). The basic inputs are three-point amplitudes, plus the three-point MHV and anti-MHV form factors,

$$F_3^{\text{MHV}}(i_\phi, j_\phi, k_{g^+}) = \frac{\langle i j \rangle}{\langle j k \rangle \langle k i \rangle}, \quad F_3^{\overline{\text{MHV}}}(i_\phi, j_\phi, k_{g^-}) = \frac{[i j]}{[j k][k i]}, \quad (\text{A.1})$$

which can be easily derived by using Feynman diagrams.

As is standard in the BCFW recursion relation, we use an $[i, j]$ shift, $\tilde{\lambda}_i \rightarrow \tilde{\lambda}_i - z\tilde{\lambda}_j$, $\lambda_j \rightarrow \lambda_j + z\lambda_i$. Thus we obtain a one complex parameter family of form factors, $F(z)$. As mentioned earlier, factorisation theorems are also valid for form factors; therefore, exactly as in the case of scattering amplitudes, by using Cauchy's theorem we can calculate the form factor by summing the residue of the poles from various factorisation channels (we also require $F(z) \rightarrow 0$ as $z \rightarrow \infty$),

$$F(0) = \sum_{a,b,h} F_L^h(z=z_{ab}) \frac{1}{P_{ab}^2} A_R^{-h}(z=z_{ab}) + \sum_{c,d,h} A_L^h(z=z_{cd}) \frac{1}{P_{cd}^2} F_R^{-h}(z=z_{cd}), \quad (\text{A.2})$$

as shown in Figure 10. Notice that there are two diagrams, since we can insert the operator either on the left- or on the right-hand side.

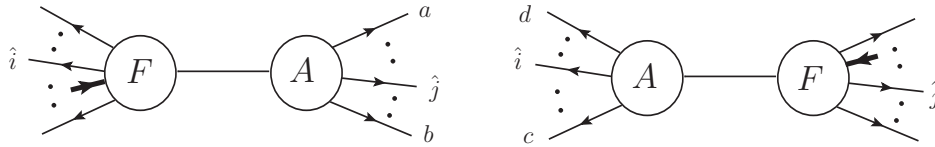


Figure 10: *BCFW recursion diagrams for tree-level form factors.*

As an example, we will derive the expression for the MHV form factor. Since we require the form factor $F(z)$ to vanish as $z \rightarrow \infty$, we shift the momenta of two positive-helicity gluons. It is not difficult to see that the non-vanishing diagrams are those where the right-hand side is an anti-MHV three-point amplitude. Without loss of generality, we calculate the n -point form factor $F_n(1_\phi, 2^+, 3_\phi, 4^+, \dots, n^+)$ by

performing a $[2, 4]$ shift. There are two non-vanishing channels to consider. The contribution of the first one is

$$\begin{aligned} F_{n-1}(5^+, \dots, n^+, 1_{\phi_{12}}, 2^+, \hat{P}_{\phi_{12}}) & \frac{1}{s_{34}} A_3(\hat{P}_{\phi_{34}}, 3_{\phi_{12}}, 4^+) \\ &= \frac{\langle 1 \hat{P} \rangle^2}{\langle 5 6 \rangle \dots \langle 2 \hat{P} \rangle \langle \hat{P} 5 \rangle} \frac{1}{s_{34}} \frac{[3 4][4 \hat{P}]}{[\hat{P} 3]}, \end{aligned} \quad (\text{A.3})$$

where $\hat{P} = P_{34} + z_{34}\lambda_2\tilde{\lambda}_4$, $z_{34} = -\langle 4 3 \rangle / \langle 2 3 \rangle$. The other one is

$$\begin{aligned} F_{n-1}(6^+, \dots, 1_{\phi_{12}}, 2^+, 3_{\phi_{12}}, \hat{P}'^+) & \frac{1}{s_{45}} A_3(\hat{P}'^-, 4^+, 5^+) \\ &= \frac{\langle 1 3 \rangle^2}{\langle 6 7 \rangle \dots \langle 3 \hat{P}' \rangle \langle \hat{P}' 6 \rangle} \frac{1}{s_{45}} \frac{[4 5]^3}{[\hat{P}' 4][5 \hat{P}']}, \end{aligned} \quad (\text{A.4})$$

where $\hat{P}' = P_{45} + z_{45}\lambda_2\tilde{\lambda}_4$, $z_{45} = -\langle 4 5 \rangle / \langle 2 5 \rangle$. The sum of these two terms indeed reproduces the MHV form factor result (3.3). Notice that as an illustration we chose a complicated shift; it would be much simpler to shift the momenta of two adjacent gluons.

B The iterative structure of [9] for the two-loop form factor

In [9], Van Neerven calculated the one- and two-loop contribution to the Sudakov form factor $F(q^2, \epsilon)$, defined in (3.1). His results are¹⁰

$$F^{(1)}(q^2, \epsilon) = 2 \text{Tri}(q^2, \epsilon), \quad (\text{B.1})$$

$$F^{(2)}(q^2, \epsilon) = 4 \text{LT}(q^2, \epsilon) + \text{CT}(q^2, \epsilon), \quad (\text{B.2})$$

where the one-loop one-mass triangle $\text{Tri}(q^2, \epsilon)$ and the two-loop ladder and crossed triangle, $\text{LT}(q^2, \epsilon)$, $\text{CT}(q^2, \epsilon)$ respectively, are given by¹¹

$$\begin{aligned} \text{Tri}(q^2, \epsilon) &= -(-q^2)^{-\epsilon} e^{\epsilon\gamma} \frac{\Gamma(1+\epsilon)\Gamma^2(-\epsilon)}{\Gamma(1-2\epsilon)} \\ &= (-q^2)^{-\epsilon} \left[-\frac{1}{\epsilon^2} + \frac{\zeta_2}{2} + \frac{7}{3}\zeta_3\epsilon + \frac{47}{1440}\pi^4\epsilon^2 + \mathcal{O}(\epsilon^3) \right], \end{aligned} \quad (\text{B.3})$$

¹⁰In this appendix we indicate explicitly the dependence of the form factor on the dimensional regularisation parameter ϵ .

¹¹In the following formulae we actually divide these functions by a power of q^2 per loop.

$$\begin{aligned} \text{LT}(q^2, \epsilon) &= (-q^2)^{-2\epsilon} e^{2\gamma\epsilon} \left\{ \frac{1}{\epsilon} \left[\frac{1}{2\epsilon} G(2, 2) G_3(2 + \epsilon, 1, 1) \right. \right. \\ &\quad \left. \left. - G(2, 1) \left[\frac{1}{\epsilon} G_3(2, 1, 1 + \epsilon) + G_3(1, 1, 1) \right] \right] \right\} \end{aligned} \quad (\text{B.4})$$

$$\begin{aligned} &= (-q^2)^{-2\epsilon} \left[\frac{1}{4\epsilon^4} + \frac{5\pi^2}{24\epsilon^2} + \frac{29}{6\epsilon} \zeta_3 + \frac{3}{32} \pi^4 + \mathcal{O}(\epsilon) \right] , \\ \text{CT}(q^2, \epsilon) &= (-q^2)^{-2\epsilon} \left[\frac{1}{\epsilon^4} - \frac{\pi^2}{\epsilon^2} - \frac{83}{3\epsilon} \zeta_3 - \frac{59}{120} \pi^4 + \mathcal{O}(\epsilon) \right] , \end{aligned} \quad (\text{B.5})$$

where

$$G(x, y) = \frac{\Gamma(x + y + \epsilon - 2) \Gamma(2 - \epsilon - x) \Gamma(2 - \epsilon - y)}{\Gamma(x) \Gamma(y) \Gamma(4 - x - y - 2\epsilon)} , \quad (\text{B.6})$$

$$G_3(x, y, z) = \frac{\Gamma(2 - x - z - \epsilon) \Gamma(2 - y - z - \epsilon) \Gamma(-2 + x + y + z + \epsilon)}{\Gamma(x) \Gamma(y) \Gamma(4 - x - y - z - 2\epsilon)} . \quad (\text{B.7})$$

In [10], the functions $\text{LT}(q^2, \epsilon)$ and $\text{CT}(q^2, \epsilon)$ were derived from dimensionally-regulated dispersion integrals; here we present these functions in the form given in [52].

In [9] it was proved that the Sudakov form factor exponentiates at two loops. In formulae, one finds

$$F^{(2)}(q^2, \epsilon) - \frac{1}{2} \left(F^{(1)}(q^2, \epsilon) \right)^2 = (-q^2)^{-2\epsilon} \left[\frac{\zeta_2}{\epsilon^2} + \frac{\zeta_3}{\epsilon} + \mathcal{O}(\epsilon) \right] . \quad (\text{B.8})$$

We can recast (B.8) in an ABDK/BDS form, namely

$$F^{(2)}(q^2, \epsilon) - \frac{1}{2} \left(F^{(1)}(q^2, \epsilon) \right)^2 = f_{\text{FF}}^{(2)}(\epsilon) F^{(1)}(q^2, 2\epsilon) + C_{\text{FF}}^{(2)} + \mathcal{O}(\epsilon) , \quad (\text{B.9})$$

where $f_{\text{FF}}^{(2)}(\epsilon) = \tilde{f}_0 + \tilde{f}_1 \epsilon + \tilde{f}_2 \epsilon^2$. We then find

$$\tilde{f}_0 = -2\zeta_2 , \quad \tilde{f}_1 = -2\zeta_3 . \quad (\text{B.10})$$

We also find a condition relating f_2 and $C_{\text{FF}}^{(2)}$, namely

$$C_{\text{FF}}^{(2)} = \frac{\tilde{f}_2}{2} + \frac{\pi^4}{18} . \quad (\text{B.11})$$

On the other hand, the four-point MHV amplitude (divided by the tree-level amplitude) satisfies [11, 12]

$$M^{(2)}(\epsilon) - \frac{1}{2} \left(M^{(1)}(\epsilon) \right)^2 = f^{(2)}(\epsilon) M^{(1)}(2\epsilon) + C^{(2)} + \mathcal{O}(\epsilon) , \quad (\text{B.12})$$

with $f^{(2)}(\epsilon) = f_0 + \epsilon f_1 + \epsilon^2 f_2$ and

$$f_0 = -\zeta_2 = \frac{\tilde{f}_0}{2} , \quad f_1 = -\zeta_3 = \frac{\tilde{f}_1}{2} . \quad (\text{B.13})$$

The factor of $1/2$ in the result (B.13) is a matter of convention – it can be understood once one recalls that f_0 and f_1 are written in a convention where the 't Hooft coupling a_{BDS} is twice as that used in the present paper (as well as in [9]). Indeed, inspecting (B.12) one quickly realises that the combination $af^{(2)}$ must be independent of any conventions used to define the coupling, since the left-hand side $a^2M^{(2)}$ is clearly convention independent.

C The two-mass easy box function

A compact form of the finite part of a two-mass easy box function containing only four dilogarithms was first derived in [53], and then found independently in [22] in the context of MHV diagrams, where an analytic proof of its equivalence with the conventional expression of e.g. [54] was given. Expressing the two-mass easy box as a function of the kinematic invariants $s := (P + p)^2$, $t := (P + q)^2$ and P^2 , Q^2 , with $p + q + P + Q = 0$, its finite part is

$$\text{Fin}^{2\text{me}}(s, t, P^2, Q^2) = \text{Li}_2(1 - aP^2) + \text{Li}_2(1 - aQ^2) - \text{Li}_2(1 - as) - \text{Li}_2(1 - at) , \quad (\text{C.1})$$

where

$$a = \frac{P^2 + Q^2 - s - t}{P^2Q^2 - st} . \quad (\text{C.2})$$

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